Algebraic Logic Error in Bell's (1964) Theorem – The Very Short Version

by Richard P. Cember
Takoma Park, Maryland, U. S.
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rcember.crbt@gmail.com

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and also on the ResearchGate page of the author.

Abstract

The derivation of Bell's Inequality in Bell (1964; hereafter, B64) depends critically on the use of the equation $A(\mathbf{a},\lambda) = -B(\mathbf{a},\lambda)$. This equation corresponds to the case in which the two detectors A and B are oriented parallel to each other. In B64, the inequality is not derived for the more general case, in which detectors A and B are oriented neither parallel nor anti-parallel to each other. The inequality therefore does not address this case, which is the case of principal interest. Contrary to the conclusion of B64, Bell's Inequality tells us nothing, one way or the other, about the potential status in quantum theory of hidden variables, local or otherwise, or of non-locality ("spooky action").

Introduction

As discussed in my recent preprint (Cember, 2020), the mathematical-physical argument that is today called Bell's Theorem (Bell, 1964; hereafter, B64), is not a definitive proof of non-locality. This is due, among other reasons, to the presence of two independent errors in its algebraic logic, either of which alone is fatal to the argument. The purpose of this short paper is to explain one of

[1] The revision consists solely of the addition of an abstract. The correction is made to one of the references.

the two algebraic logic errors as concisely as possible. It is not the purpose of the present paper to argue for or against non-locality.

It is assumed here that the reader is familiar with B64. Thus, no background material is presented. References are kept to a minimum. The symbols used in this paper are the same as those used in B64, with the same meanings assigned to them. Equation numbers are the same as those used in B64.[3] This paper is a short version of the material presented in Section 3 of Cember (2021). For a fuller discussion of this and other material, including the other, independent, fatal algebraic logic error, please see Cember (2020).

Dependence of the derivation of Bell's Inequality on its algebraic assumptions[4]

In the mathematical symbolism used by B64 to describe the thought experiment of Bohm (1951), there are three possible cases of the relationship between the respective results A and B, obtained for one pair of particles in the two detectors A and B, as a function of the orientations of the unit vectors **a** and **b**. These are

Case I: when $\mathbf{a} \cdot \mathbf{b} = 1$, $A(\mathbf{a}, \lambda) = -B(\mathbf{b}, \lambda)$
Case II: when $\mathbf{a} \cdot \mathbf{b} = -1$, $A(\mathbf{a}, \lambda) = B(\mathbf{b}, \lambda)$
Case III: when $|\mathbf{a} \cdot \mathbf{b}| \neq 1$, $A(\mathbf{a}, \lambda) = B(\mathbf{b}, \lambda)$ or[5] $A(\mathbf{a}, \lambda) = -B(\mathbf{b}, \lambda)$

These three cases (not identified by these names in B64) reflect the quantum theory of the Bohm experiment. They are mutually exclusive cases and they collectively exhaust all possibilities for the experimental result for one pair of particles.

Case I corresponds to Equation (13) in B64:

$$A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda) \ . \quad (13)$$

Equation (2) of B64 is

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda \, \rho(\lambda) \; A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \ . \quad (2)$$

[3] B64 is reprinted also in Bell (1987). For the reader of this paper, an adequate substitute for the original B64 paper is Chapter 12, Section 2 of Griffiths (2005), which uses the same symbols as B64 and very nearly the same mathematical expressions, while eliminating nonessential material. However, Griffiths' equation numbers are not the same as those used in B64 and here.
[4] I focus here only on the algebraic assumptions. There is a good deal of controversy, among both its supporters and its detractors, about what the other assumptions of B64, stated and unstated, actually are. That controversy does not affect the argument given here.
[5] This is an exclusive "or", *i.e.*, in Case III, the product AB may be either +1 or -1. A and B are unspecified single-valued functions of their arguments.

Equation (13) is substituted into Equation (2) to obtain Equation (14):

$$P(\mathbf{a},\mathbf{b}) = -\int d\lambda\, \rho(\lambda)\ A(\mathbf{a},\lambda)\, A(\mathbf{b},\lambda)\ . \quad (14)$$

Equation (14) may be written in terms of an alternate setting of one of the detectors, setting it to **c** rather than to **b**:

$$P(\mathbf{a},\mathbf{c}) = -\int d\lambda\, \rho(\lambda)\ A(\mathbf{a},\lambda)\, A(\mathbf{c},\lambda)\ .$$

The preceding equation is then subtracted from Equation (14) to obtain the following unnumbered equation, which I will call Equation (A):

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = -\int d\lambda\, \rho(\lambda)\ A(\mathbf{a},\lambda)\, A(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)\, A(\mathbf{c},\lambda)\ . \quad \text{(A)}$$

The next manipulation is to factor the integrand of Equation (A) to obtain another unnumbered equation, which I will call Equation (B):

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = \int d\lambda\, \rho(\lambda)\ A(\mathbf{a},\lambda)\, A(\mathbf{b},\lambda)[\, A(\mathbf{b},\lambda)\, A(\mathbf{c},\lambda) - 1\,]\ . \quad \text{(B)}$$

After further manipulation, including application of the absolute value operator, Equation (15), known today as Bell's Inequality, is obtained:

$$1 + P(\mathbf{b},\mathbf{c}) \ \geq |\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})\, |\ . \quad (15)$$

Equation (15) is derived for pairs of detector settings, *i.e.*, experimental configurations, selected only from Case I. For $P(\mathbf{a}, \mathbf{b})$, the quantum theory gives us

$$P(\mathbf{a},\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b}\ .$$

Not surprisingly, then, for $\mathbf{a} = \mathbf{b} = \mathbf{c}$, for which Case I obtains for all three of the unordered pairs of unit vectors that can be selected from the set **a**, **b**, **c**[6], Bell's Inequality, Equation (15), is satisfied by values of $P(\mathbf{a}, \mathbf{b})$ computed from the quantum theory.

In analogy to Equation (13), Case II may be written as

$$A(\mathbf{a},\lambda) = B(-\mathbf{a},\lambda)\ , \qquad \text{(C)}$$

[6] The possible unordered pairs are (**a**, **b**), (**a**, **c**), (**b**, **c**).

which I will call Equation (C). Equation (C) does not explicitly appear in B64. The derivation of Equation (15) in B64 takes no explicit account of experimental configurations that include Case II. Properly speaking, Equation (15) in B64 does not address Case II. It is not possible for Case II to hold at once for all three of the unordered pairs of unit vectors that may be selected from the set **a**, **b**, **c**. However, although it is not shown in B64, using Equations (13) and (C), Equation (15) can be derived for either of the relations $\mathbf{a} = -\,\mathbf{b} = \mathbf{c}$ or $\mathbf{a} = \mathbf{b} = -\,\mathbf{c}$, by a method similar to that used in B64. These two relations among **a**, **b** and **c**[7] are each a combination of Case I and Case II. For either of these relations among **a**, **b** and **c**, Bell's Inequality, Equation (15), is satisfied by values of $P(\mathbf{a}, \mathbf{b})$ computed from the quantum theory.

Thus, for Case I, and also for a combination of Case I and Case II, Bell's Inequality is compatible with the quantum theory of the Bohm experiment.

Equation (15) as derived in B64 takes no account of Case III. For Case III, the factorization that makes Equation (A) into Equation (B) is impossible, because neither Equation (13) nor Equation (C) may be used. Thus, for experimental configurations involving Case III, Bell's Inequality, as derived in B64, makes no algebraic assertion about the results of experiment for Case III. It is irrelevant to Case III.

The counter-example

Recall that for $P(\mathbf{a}, \mathbf{b})$, the quantum theory gives $P(\mathbf{a}, \mathbf{b}) = -\,\mathbf{a} \cdot \mathbf{b}$. The counter-example that is chosen in B64 to show that Bell's Inequality is incompatible with the quantum theory is

$$\mathbf{a} \cdot \mathbf{c} = 0 \quad \text{and} \quad \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = 1/\sqrt{2} \ .$$

This example may be visualized as two mutually orthogonal unit vectors **a** and **c**, with **b** as a third unit vector lying in the common plane of **a** and **c** at an angle of 45° from each. In this case, Equation (15) becomes

$$1 - \cos 45^{\circ} \quad \geq \quad |-\cos 45^{\circ} \quad - \; 0 \,| \ ,$$

or

$$0.293 \ \geq \ 0.707 \ ,$$

which is obviously false. This counter-example is offered in B64 to show that the use of the variable λ to represent any hidden variable leads to a contradiction with the quantum theory of the Bohm experiment.

[7] That is, $\mathbf{a} = -\,\mathbf{b} = \mathbf{c}$ and $\mathbf{a} = \mathbf{b} = -\mathbf{c}$.

More generally, it may be noted that for $\mathbf{a} \cdot \mathbf{c} = 0$, Equation (15) is false for any unit vector **b** lying in the quadrant of the plane between **a** and **c**. There are thus many "counter-examples" available. However, all such "counter-examples," including the "counter-example" in B64, are drawn from Case III, the case that Bell's Inequality, does not cover. Thus they are not "counter-examples" at all. Rather, they are observations that Equation (15) does not hold in the domain for which it was not derived; a domain which is, moreover, disjoint from the domain for which it *was* derived. This observation ought not to be seen as surprising or enlightening. The application of Bell's Inequality to Case III, for which it was not derived, is a fatal error of algebraic logic in B64.

Conclusions

Given the unsurprising fact that B64's distinctive scheme of unspecified functions and variables A, B, λ and $\rho(\lambda)$, related by Equation (13), does not work for Case III, are we to draw the conclusion that some alternative scheme that *would* work for Case III, cannot possibly exist? The answer is: No, that simply does not follow.

Final remark

From the failure of B64 as a proof of non-locality, it does not follow that non-locality as a proposition must be false. It only means that B64 fails to throw any additional light, relative to what was already in the literature before its publication, on the question of what the truth value of the proposition of non-locality actually is.

There may be reasons other than B64 to embrace non-locality. One reason, of course, is the very analysis of quantum mechanics made by Einstein, Podolsky and Rosen (1935), although they themselves rejected non-locality because of its implications. Many other authors have put forward additional arguments for non-locality, including, though not limited to, other ways of developing Equation (15). Those other reasons and other arguments are not evaluated here.

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