Notational Inconsistency in Bell's (1964) Theorem – The Very Short Version

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Abstract

In the derivation of Bell's Inequality in Bell (1964) (hereafter, B64), hypothetical local hidden variables ("λ") are treated algebraically as if they have a single, non-local (global) value. This is in contradiction to the stated premise of the paper, with a result at variance to the stated conclusion of the paper. Bell's statement justifying this non-intuitive notational approach ("Some might prefer…") does not withstand examination.

Introduction

As discussed in my recent preprint (Cember, 2020), the mathematical-physical argument that is today called Bell's Theorem (Bell, 1964; hereafter, B64), is not a definitive proof of non-locality. This is due, among other reasons, to the presence of two independent errors in its algebraic logic, either of which alone is fatal to the argument. A previous "very short version" article (Cember, 2022) concisely discussed one of these two independent fatal errors. The purpose of the present "very short version" article is to explain the second of the two algebraic errors as concisely as possible. The second error, the one to be discussed here, has the effect of algebraically treating local hidden variables as if they have a single, non-local (global) value. [2]

It is not the purpose of the present paper to argue for or against non-locality. Rather, the purpose is to exhibit one reason why Bell's Theorem is not a proof of non-locality.

I focus here solely on an algebraic issue. There is a good deal of controversy, among both its supporters and its detractors, about what the other assumptions of B64, stated and unstated, actually are. That controversy does not affect the argument given here.

It is assumed here that the reader is already quite familiar with B64 (although such readers might find it useful to have a copy of B64 at hand). Thus, no background material is presented. References are kept to a minimum. The symbols used in this paper are the same as those used in B64, with the same meanings assigned to them. Equation numbers are the same as those used in B64.[3] This article is a version of the material presented in Section 4 of Cember (2020). For a fuller discussion of this and other material, please see Cember (2020).

The problematic statements

In Section II of B64, Bell writes,

> Let this more complete specification [by hidden variables] be effected by means of parameters λ. It is a matter of indifference in the following whether λ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous. However, we write as if λ were a single continuous parameter.

Later, Bell continues:

> Some might prefer a formulation in which the hidden variables fall into two sets, with A dependent on one and B dependent on the other; this possibility is contained in the above, since λ stands for any number of variables and the dependence thereon of A and B are unrestricted.

These statements imply that the two formulations are equivalent, leading to the same results and the same conclusions. It will be shown here that this is not true.

[2] I think it is useful and clarifying here to borrow the term "global" from the domain of computer programming.
[3] B64 is reprinted also in Bell (1987). For the reader of this paper, an adequate substitute for the original B64 paper is Chapter 12, Section 2 of Griffiths (2005), which uses the same symbols as B64 and very nearly the same mathematical expressions, while eliminating nonessential material. However, Griffiths' equation numbers are not the same as those used in B64 and here.

Trying the formulation that some might prefer

Let us try the formulation that "some might prefer," to see what effect it has on the result.

We begin with a modification of Equation (2) of B64. In the following, prime symbols (′) are used on equation numbers to indicate correspondence, albeit with a difference, to similarly numbered equations in B64.

$$P(\mathbf{a},\mathbf{b}) = \int\int \rho(\lambda_1,\ \lambda_2)\ A(\mathbf{a},\lambda_1)\ B(\mathbf{b},\lambda_2)\ d\lambda_1\ d\lambda_2\ . \quad (2')$$ [4]

Here, we have assumed that the local hidden variable λ_i is associated with the *i*th particle and not with the detector through which it passes. This is the case of principal interest. Later we will also briefly consider the case in which the local hidden variable may be associated with the detector, or with both the particle and the detector. For definiteness, and without loss of generality, we identify the index 1 with the particle that transits detector A.

We must also modify Equation (13) of B64 to become (13′):

$$A(\mathbf{a},\lambda_1) = -B(\mathbf{a},\lambda_2)\ . \quad (13')$$

A clearer way to understand (13′) is to read it as

$$A(\mathbf{n},\lambda_1) = -B(\mathbf{n},\lambda_2)\ , \quad (13\mathrm{a}')$$

where **n** represents any unit vector that may be used as a setting for a detector.[5]

With (13a′) in mind, (13′) is inserted into (2′) in the manner of B64, leading to a modified version of B64's Equation (14)[6]:

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[4] The density function $\rho\,(\lambda_1,\ \lambda_2)$ in Equation (2') includes the special case of $\rho = \rho_1(\lambda_1)\rho_2(\lambda_2)$, but this case is not of interest in this context.

[5] The way in which B64 uses the symbols *A*, *B*, **a**, **b** and **c** can be very confusing (*A* vs. **a**, *B* vs. **b, c** as the alternate setting of **b**, etc.). This is another very poor choice of notation in B64, though if used with sufficient care it need not lead to error. The key to the reduction of this potential notational confusion is the judicious mental use of a symbol such as **n**, rather than **a**, **b** or **c**, to refer to a unit vector, when doing so provides greater clarity. With this approach, Appendix 3 of Cember (2020) provides the derivation of Bell's Inequality using a less confusing set of symbols.

[6] Sharp-eyed readers may notice that in Cember (2020) I myself made an index error in the equation there, (7′′), that is similar to (14′) here. It turns out (and readers who want to take the necessary time and trouble can verify this) that the conclusions of that work are not affected.

$$P(\mathbf{a},\mathbf{b}) = -\int\int \rho(\lambda_1, \lambda_2)\, A(\mathbf{a},\lambda_1)A(\mathbf{b},\lambda_1)\ d\lambda_1\, d\lambda_2\,. \quad (14')$$

We now follow the algebraic steps of B64 and derive an equation corresponding to an unnumbered equation in B64 that immediately precedes B64's Equation (15). Here, for identification, we will call that un-numbered equation (14c′), as it corresponds to the third equation appearing in B64 after Equation (14).

$$|\,P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq \int\int \rho(\lambda_1, \lambda_2)\, d\lambda_2\, [1 - A(\mathbf{b},\lambda_1)A(\mathbf{c},\lambda_1)]\ d\lambda_1\,. \quad (14c')$$

This equation may be simplified and identified here as (14d′):

$$|\,P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq \int \rho(\lambda_1)\ [1 - A(\mathbf{b},\lambda_1)A(\mathbf{c},\lambda_1)]\, d\lambda_1\,. \quad (14d')$$

Equation (14d′) is then rearranged for easy comparison to B64's Equation (15), Bell's Inequality, as

$$1 + \int -\rho(\lambda_1)\ A(\mathbf{b},\lambda_1)A(\mathbf{c},\lambda_1)\, d\lambda_1\ \geq |\,P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})|\ . \quad (15')$$

Now we see that we encounter a difficulty. There is no way to perform any experiments in such a way that both detectors are exposed *only* to the local variable value (here, λ_1) associated with only *one* of the particles. Therefore, we cannot use a large number of experiments to make an empirical estimate of the value of the integral. Also, we know nothing about the dependence of A on λ_1. Thus we also cannot theoretically predict the value of the integral. It follows then that we cannot compare (15′) either to theory or to experiment, and we can therefore infer nothing at all about its truth value. Indeed, it is not even clear what proposition Equation (15′) might represent.

Equation (15′) therefore does not contribute a falsifiable statement – or indeed, any clearly identifiable statement – that is relevant to consideration of the implications of hypothetical local hidden variables. But (15′) is what we end up with when we attempt to treat *local* hidden variables in an intuitively natural and algebraically consistent manner, that is to say, in the manner that "some might prefer."

On the other hand, if there is only a single non-local (global) λ, such that $\lambda_1 = \lambda_2 = \lambda$, then (15′) is reduced to B64's Equation (15), which is Bell's Inequality. Bell's Inequality can be, and of course has been, compared to the quantum theory of the Bohm (1951) thought experiment, as well as, in the form of the CHSH Inequality (Clauser, Horne, Shimony and Holt, 1969), to the results of experimental realizations of the Bohm experiment.

We see that if we consider the way the variable λ is actually employed in the derivation of Bell's Inequality, where λ is algebraically treated as if its value is identically the same in the arguments of functions *A* and *B*, then the inequality would seem to exclude hypothetical *non-local (global)* hidden variables, while saying nothing about the case of hypothetical *local* hidden variables. This is the exact opposite of the intended, and stated, conclusion of B64.[7]

If we were to associate the local hidden variable with the detector A instead of with the particle, or with both the detector and the particle, then $A(\mathbf{c}, \lambda_1)$ would become $A(\mathbf{c}, \lambda_A)$, or $A(\mathbf{c}, \lambda_{A,} \lambda_1)$, or some other expression for *A* as a function of a more complicated set of arguments relating the local variables to their local environments. The impossibility of comparing an inequality similar to (15′) to either theory or experiment would not be relieved.

What, then, of Bell's assertion that

> "...a formulation in which the hidden variables fall into two sets, with *A* dependent on one and *B* dependent on the other...is contained in the above, since λ stands for any number of variables and the dependence thereon of *A* and *B* are unrestricted"?

This assertion seems to amount to the suggestion that λ is to be understood as a vector $\lambda = (\lambda_1, \lambda_2)$[8] (indeed, B64 says this much); that $A(\mathbf{n}, \lambda)$ is to be understood in the sense of $A(\mathbf{n}, \lambda_1, \lambda_2)$, and similarly for *B*; that, however, λ_2 is to be understood as an argument of *A* only in a purely formal sense, but not having any actual effect on *A*; while λ_1 is to be understood as a correspondingly purely formal argument of *B*, lacking any actual effect on *B*. This notational approach of B64 has resulted in confusion and logical error. Those who might have preferred the more explicit formulation used in Equation (2′) would have been right to do so.

Final remark

If the hypothetical variables are to be considered as local, then their values must differ between the spatial locations in which they are to have their hypothetical effect. They cannot then be represented in algebraic manipulations by a single non-local (global) symbol. When such an effectively global symbol is used, conclusions drawn from algebraic manipulations using that *global* symbol are not relevant to *local* variables.

References

[7] But of course, we could not actually draw this conclusion, either, because the derivation of Bell's Inequality in B64 is inconsistent for additional reasons completely independent of the discussion here, as explained in Cember (2020) and Cember (2022). But those considerations are outside the argument of this article.

[8] To be more precise, it is the suggestion that $\lambda = (\lambda_1, \lambda_{2, \ldots,} \lambda_N)$, where there may be any number N of components λ_i. For our purposes, N=2 is sufficient.

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