# A Critical Re-examination of Bell's Theorem

by

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Slightly revised and corrected 22 November 2020,
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The following changes were made in the revision of 22 November 2020, relative to that of 4 July 2020:

1. Known typographical or syntactical errors were corrected.

2. The reference to Sanctuary (2006) was added.

3. A few words were added or deleted to improve clarity.

4. Footnote 14 was rewritten for clarity.

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1. A few sentences were added to the abstract.

2. Some italics were removed in Section 3.

3. A few words and sentences were added, removed, or moved to improve clarity.

4. The references to Christian (2018) and Muchowski (2020) were added.

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1. A few words and sentences were added, removed, or moved to improve clarity.

2. Figure 2 was deleted as unnecessary.

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*Abstract*

Bell's Theorem (Bell, 1964; hereafter, B64) is widely considered to be a definitive proof of "non-locality," or, as Einstein called it, "spooky action at a distance." In the present article it is shown that the derivation of Bell's Inequality in B64 contains mathematical errors, and that therefore Bell's Theorem is not a proof of non-locality. In this respect the present article is a critique of Bell's Theorem principally on grounds of internal consistency; it does not attempt directly to resolve the foundational problem of non-locality. Moreover, it is argued here that even if Bell's Theorem did not contain mathematical errors, it would still not be a definitive proof of non-locality because it addresses only deterministic hidden variables, a category which is not comprehensive of all potentially reasonable hidden-variable theories.

Bell's Theorem is an analysis of Bohm's (1951a) thought experiment, which concerns two spin-½ particles in the singlet state that become separated and move in opposite directions. The nominal program of B64 is as follows. To the assumptions of the quantum theory of the Bohm thought experiment, B64 adds additional assumptions concerning hypothetical local hidden variables. Using these additional assumptions, B64 derives an inequality (Bell's Inequality) describing the predicted results of the experiment. The inequality is then compared to the predictions of the quantum theory. The derived inequality fails to agree with the quantum-theoretical prediction. The inequality also fails to agree with the results of experimental realizations of the Bohm thought experiment, which, with one accidental exception, occurred later than the publication of B64. The failure of the inequality to agree with the predictions of the quantum theory and with the results of experimental realizations − which agree with each other − is taken to refute the additional assumptions of B64 concerning local hidden variables, and therefore to constitute a proof of non-locality (spooky action).

The argument of Bell's Theorem, however, fails in at least three independent respects.

(1) The derivation of Bell's Inequality in B64 depends critically on an inadmissible algebraic substitution.

(2) The derivation of Bell's Inequality in B64, even if it were algebraically valid, would exclude only hypothetical non-local hidden variables, not local ones. The apparent applicability of Bell's Inequality to the case of local hidden variables is the result of confusion caused by a poor choice of notation.

(3) The schema of B64 addresses only the case of hypothetical deterministic hidden variables. Determinism (though not by that name) is one of the demands of Einstein, Podolsky and Rosen (1935). B64 does not address the case of hypothetical hidden variables that influence the local probability distributions for the outcomes of observations but do not uniquely determine the outcomes. Thus, even if it were otherwise valid, Bell's Theorem would not constitute an airtight proof of non-locality simply because it does not cover this important class of cases.

The failure of Bell's Inequality to agree with experimental realizations of the Bohm experiment has been taken as incontrovertible proof of the reality of spooky action. However, theoretical predictions may fail to agree with the results of experiment for reasons that do not signify anything about the truth or falsehood of their assumptions. In particular, they may fail because of errors in the logic that carries their assumptions into their predictions. This is the case with Bell's Inequality.

There might be good reasons to accept non-locality, a.k.a. spooky action. However, if there are, Bell's Theorem is not one of them.

*Prefatory notes*

1. Potential to skip some sections. Some readers of this article will already be very familiar with this subject, others not. Because of the universal importance of the foundational questions of quantum mechanics, and the broad interest in them among physicists in general, I have tried to write this article in such a way as make it accessible to all physicists. However, the article does assume that at some point in his or her education the reader has been exposed to the foundational questions of quantum mechanics. Readers who are very familiar with the subject may wish to skip some sections or appendices, at least on a first reading; these sections and appendices are indicated by an asterisk (*).

2. Terminology. In this article I use the term "Bell's Theorem" to refer to the whole of the main line of the argument of B64, including the counter-example and the resulting conclusions, and not solely to the derivation *per se* of Bell's Inequality. When referring specifically to the derivation of the inequality and not to the argument of B64 as a whole, I use the word "derivation".

**Historical background*

Despite its great experimental success, quantum mechanics has always been plagued by what are often called its "foundational" questions or problems. One of these is the phenomenon famously described by Albert Einstein in a letter to Max Born (Born, 1971) as "spooky actions at a distance".

Einstein and Niels Bohr met for the first time in Berlin in 1920 (Bohr, 1949). At this first meeting they discussed what today we call foundational questions. A private discussion thus begun in 1920 became a friendly but vigorous public debate at the Solvay Conferences of 1927 and 1930. The thread of the debate is recorded by Bohr (1949).

In 1935, Einstein, together with his associates Boris Podolsky and Nathan Rosen, published their famous paper asserting the incompleteness of quantum mechanics, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" (Einstein, Podolsky and Rosen, 1935; hereafter, EPR35). Bohr (1949) writes:

> Due to the lucidity and apparently incontestable character of the argument, the paper of Einstein, Podolsky and Rosen created a stir among physicists and has played a large role in general philosophical discussion.

In 1936, Bohr published his rejoinder (Bohr, 1936), also now famous. Bohr (1949) continues:

> It will be seen...that we are here [in EPR35] dealing with problems of just the same kind as those raised by Einstein in previous discussions, and in an article which appeared a few months later [Bohr, 1936], I tried to show that from the point of view of complementarity the apparent inconsistencies were completely removed.

Although much has been written and said since 1936, the debate between Einstein and Bohr that crystallized in the papers of 1935 and 1936 has ever since set the essential terms of discussion with regard to the foundational problems of quantum mechanics.

By 1949 Einstein was almost alone among the most prominent physicists in rejecting Bohr's view. Pauli (1949), with his characteristic directness, writes,

> The writer [Pauli] belongs to those physicists who believe that the new epistemological situation underlying quantum mechanics [Bohr's concept of complementarity] is satisfactory, both from the standpoint of physics and from the broader standpoint of the conditions of human knowledge in general. He regrets that Einstein seems to have a different opinion on this situation; [...]

Similar sentiments were expressed, with varying degrees of greater delicacy, by others among the great twentieth-century physicists who contributed to Schilpp's (1949) collection of essays about, and by, Einstein. It is a commonplace today that time has granted Bohr a complete victory in this debate.[1]

In EPR35, Einstein, Podolsky and Rosen (EPR) argued that quantum mechanics must be incomplete. They objected to three characteristics of quantum mechanics that they claimed a complete physical theory would not have. Restating these in a convenient form, the three objections are:

1. In quantum mechanics, given a pair of non-commuting observables and a system to be observed, at most one of the observables may possess a well-defined value before measurement. EPR insisted that a complete physical theory must require all observables to have precisely defined values at all times.

2. In a complete physical theory, according to EPR, the value of any observable of a system must in principle be precisely predictable before measurement, given suitable auxiliary information such as boundary conditions, initial conditions, etc. Quantum mechanics does not provide this predictability; values to be observed are predicted only in a probabilistic sense.

3. In a physical system consisting of separated parts, EPR asserted that no complete physical theory – indeed, "no reasonable theory" − could allow an observation on one part of the system to influence instantaneously a subsequent measurement on the other part of the system at an arbitrary distance. The formalism of quantum mechanics seems not only to allow this instantaneous action at a distance ("spooky action"), but positively to require it.

[1] However, for a view from a different angle, see the report by Sivasundaram and Nielsen (2016) of a survey of the beliefs and opinions of contemporary physicists concerning the foundational questions. Individual physicists are anything but clear in their own minds about these matters.

For a shorthand, we may refer to these objectionable (to Einstein) characteristics as, respectively, non-commuting observables, unpredictability and spooky action.

At around this time, or somewhat earlier, Schrödinger coined the term "entanglement" to describe the phenomena, apparently suggested by the quantum theory, that gave rise to the complaints of EPR35 (Schrödinger, 1935). From the historical context it seems probable that Schrödinger intended the term to be an ironic and evocative expression of the riddle of quantum correlations, not a literal answer to the riddle.

EPR35 contained a thought experiment. The thought experiment of EPR35 considered the position and momentum of "...two systems, I and II, which we permit to interact from the time $t = 0$ to $t = T$, after which time we suppose that there is no longer any interaction between the two parts."

Bohm (1951a, hereafter B51A; see Sec. 22-16, p. 614) recast the EPR thought experiment concerning position and momentum into a thought experiment concerning the spins of two spin-½ particles originating as a single system with zero total spin (the singlet state[2]), which are then separated in such a way that the two parts can move off to a significant distance from each other. As an example of such a system, Bohm (see the next section) uses an unspecified molecule that disintegrates into two parts through some process that does not change the total angular momentum of the system.

Bohm wrote that the reason he recast the EPR thought experiment concerning position and momentum into a thought experiment concerning spin that would be "conceptually equivalent" was because it would be "considerably easier to treat mathematically", as well as probably easier to realize physically at some time in the future. "Unfortunately," Bohm wrote, "such an experiment is still far beyond present techniques, but it is quite possible that it could someday be carried out" (B51A, Sec. 22-19, p. 623). The principal reason why Bohm felt that the recast thought experiment would be both easier to discuss and easier eventually to realize was that the recasting of the experiment turned a problem of continuous observables into a problem of discrete observables. In an appendix to a later paper, Bohm and Aharonov (1957) explain in greater detail why the original position and momentum experiment of EPR35 would be, in principle, extremely difficult to accomplish.

In his book, Bohm used the recast thought experiment to show that quantum-mechanical theory could not allow the separated particles to take on definite spin values prior to the measurement of their spins, i.e., that such an assumption would lead to an inconsistency in the theory. This was the principal purpose of his discussion. Although allowing that it could not be definitively declared to be impossible, Bohm concluded that it was extremely unlikely that a theory of "hidden variables" (i.e., as-yet unknown properties and processes[3]) that could render the values

[2] For a brief review of the singlet and triplet states, see Appendix 1.
[3] More expansively, "hidden variables" are hypothetical, currently unknown properties or processes that uniquely determine the selection, when an observation is made, of a particular eigenvalue as the outcome of the

of non-commuting observables to be all well-defined prior to any measurement would ever be found, because, he said, it would require quantum mechanics to be wrong. Even some seventy years ago this was considered to be an exceedingly unlikely prospect. Bohm touched on the topic of hidden variables several times in the course of a long book (B51A), but neither made nor discussed any conceptual distinction between local and non-local hidden variables.

Bohm and Aharonov (1957) proposed that the polarizations of a pair of correlated photons could be used experimentally as a conceptually equivalent substitute for the spins of a pair of spin-½ particles as originally envisaged in the thought experiment of B51A. Bohm and Aharonov discussed an earlier experiment (Wu and Shaknov, 1950) conducted for the purpose of studying photon correlation. They showed that the experiment of Wu and Shaknov could be recognized retrospectively as a fortuitous realization of the Bohm thought experiment, and that, moreover, its results conformed to quantum-theoretical predictions but did not conform to some specific alternative hypotheses.

Curiously, notwithstanding his argument in B51A against the likelihood that any hidden variable theory could exist in conformity with quantum mechanics, Bohm (1951b, 1951c) proposed a hidden-variable theory. Although Bohm's (1951b, 1951c) "suggested interpretation" of quantum mechanics has not been widely accepted[4], research, analysis and discussion of his approach continue.[5] As these two papers of Bohm do not discuss spin, they are tangential to the present article and will not be further discussed here.

B64 (reprinted in Bell,1987) analyzed Bohm's thought experiment. Using a certain formulation to describe the hypothetical presence of local hidden variables in a Bohm-like experiment, B64 derived an inequality, now known as Bell's Inequality. The proposition in B64 is that the existence of local hidden variables would imply that the results of the Bohm experiment should satisfy Bell's Inequality; and that if the inequality is not satisfied by experiment, then local hidden variables must not exist. A corollary, about which Bell is explicit, is that if local hidden variables do not exist, then spooky action must obtain.

The derivation of Bell's Inequality is surprisingly, even shockingly, brief and simple. After the derivation, B64 shows by a very simple counter-example that the quantum theoretical predictions of the results of the Bohm experiment do not in general satisfy Bell's Inequality. Local hidden variables are thus, according to B64, proven to be incompatible with the quantum theory of the Bohm experiment, and by implication with quantum mechanics in general. This is the gist of what is now called Bell's Theorem.

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observation, when, prior to the observation, the state vector of the system offered two or more eigenvalues as possible outcomes. For systems with degenerate eigenvalues, "the outcome" is extended to include not only the eigenvalue observed but also the eigenstate in which the system is left after the observation. Hidden variables acting only in the spatial vicinity of the observation are commonly referred to as "local hidden variables".

[4] It is attributed to Wolfgang Pauli to have described Bohm's "suggested interpretation" as "a check that cannot be cashed."

[5] For example, Detlef Dürr and Stefan Teuffel, *Bohmian Mechanics: The Physics and Mathematics of Quantum Theory*, Springer-Verlag, 2009.

B64 stimulated new interest in the thought experiment of B51A, in particular in the prospect of putting quantum mechanics to rigorous experimental tests to see if the results of experiments purposely designed[6] to realize the Bohm thought experiment would conform to quantum-mechanical predictions and thus, according to B64, rule out local hidden variables. Clauser, Horne, Shimony and Holt (1969) derived directly from Bell's Inequality a straightforward extension, now known as the CHSH Inequality. The CHSH Inequality is more suitable than the original Bell's Inequality for application under realistic experimental conditions, where there is always some noise or error in the observations. In the same paper CHSH described a proposed experiment.

Photon correlation experiments seeking to realize the Bohm thought experiment ever more rigorously then followed. These used Bell's Theorem and the CHSH Inequality to explore what had come to be called the "EPR paradox"[7]. In the decades that followed the publication of B64, experiments were performed by Clauser and his colleagues (*e.g.*, Freedman and Clauser, 1972); by Aspect and his colleagues (*e.g.*, Aspect, Grangier and Roger, 1981); by Zeilinger and his colleagues (e.g, Weihs *et al.*, 1998); and by others as well. All of these experiments were recognized as *tours de force* for the experimental technology of the respective years in which they were performed. The experiments, using the polarizations of pairs of photons, tested and verified the predictions of quantum theory for realizations of the Bohm experiment under increasingly rigorous experimental conditions.

Today's textbook orthodoxy (*e.g.*, Griffiths, 2005) is that these experiments are verifications of Bell's Theorem, namely, the proposition that local hidden variables are incompatible with quantum mechanics and that therefore non-locality – spooky action – must be accepted as a feature of reality. There is also a large literature extending or considering the further implications of Bell's Theorem; for example, Greenberger, Holt, Shimony and Zeilinger (1990), Mermin (1990), Hemmick and Shakur (2018).

The ascent of Bell's Theorem to orthodoxy was not immediate. Howard and Ramirez (2019) recount the story of the quasi-journal *Epistemological Letters*, which was privately published and circulated from 1973 to 1984, during a period in which, as Howard and Ramirez describe it, study of the foundations of quantum mechanics, especially studies motivated by an interest in the implications of Bell's Theorem, did not find a forum in the regular physics journals, at least not in the United States.

Bell himself was never satisfied with the conclusions of his own famous paper. Reinhold Bertlmann, who was a close friend and colleague of Bell at CERN, the European Center for

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[6] That is to say, unlike the fortuitous realization by Wu and Shaknov (1950), discussed by Bohm and Aharonov (1957).

[7] EPR do not use the word "paradox", nor does Bohr (1936) or Furry (1936) in their respective rejoinders to EPR35. However, Bohr (1936) does use the phrase "apparent contradiction", which is a synonym for "paradox." EPR, of course, did not believe that the contradiction was merely apparent.

Nuclear Research, has written a short memoir of Bell (Bertlmann, 2015). In this memoir Bertlmann discusses Bell's feeling about the conclusions of B64. As Bertlmann describes it, Bell's view was that the apparent incompatibility of relativity with the spooky action required by B64 seemed only to leave a conundrum.

Despite the rise of Bell's Theorem to current orthodoxy, the troublesome issues associated with nonlocality remain a topic of active interest, and there is a small counter-current of critique of B64. Some examples (by no means an exhaustive list) are Sanctuary (2006), Christian (2010, 2018), Kracklauer (2015), Muchowski (2020).

At the present time, as a result of interest in potential applications of quantum entanglement in computing, communication and cryptography, the number of experiments demonstrating, exploring or exploiting aspects of entanglement is exploding. One can get a sense of the extent of this theoretical and experimental effort from the review by Flamini, Spagnolo and Sciarrino (2018) of quantum information processing using optical photons. Nearly all of the work discussed in their review, which contains over 600 references, depends in one way or another on entanglement. Moreover, as discussed by Flamini *et al.*, in addition to the optical photonics covered in their review, work is also being done in the use for quantum information processing of atomic and nuclear spins, electrons, solid state devices, Josephson junctions and superconducting devices.

*The purpose of this article*

It is the purpose of this article to show that Bell's Theorem is not a proof of spooky action because it contains mathematical and logical errors, and that even if it did not contain errors, it does not cover all important classes of cases. It is not the purpose of this article to answer any of the foundational questions of quantum mechanics.

It is impossible to discuss Bell's Theorem without reference to both spooky action and hidden variables. It is important to be clear that of those two closely related but not identical issues, it is the conclusions of B64 with respect to spooky action which principally motivate this article.

There may be good reasons to accept spooky action. Indeed, it seems from the historical context that many physicists (though not Einstein) did accept it – though not explicitly – long before the publication of B64. They did so simply because of the great experimental success of the tightly integrated linear algebraic framework ("the formalism") of quantum mechanics, which seems so strongly to suggest non-locality, though not to prove it. B64 seemed to offer a simple proof. However, it is not proof.

**The genesis of this article*

In approaching the arguments of this article, it may be helpful, although it is not essential, for readers to be aware of what my intellectual point of departure for this work was.

Probably most people with significant academic training in the physical sciences or related fields are familiar with the process of encountering a new mathematical concept, and then, in the course of studying it in school or applying it in practice, developing a “feeling” for how it works mathematically and a “sense” of why it is true. This feeling becomes a kind of unitary grasp of the whole concept. Such a grasp transcends the process of mathematical demonstration.

This mental representation, which may or may not correspond rigorously to the mathematics, generally corresponds to it well enough to guide an intuition that will successfully lead to more carefully formulated expressions and analyses. In classical branches of physics, especially in mechanics, the mathematical feeling for the concept is closely coupled to natural physical intuition. However, even in branches of study where natural physical intuition is somewhat decoupled from the mathematics, such as in quantum mechanics, one still develops a feeling for the relevant mathematical framework. In the particular case of quantum mechanics, this is a feeling for its linear algebraic framework, its “formalism.”

Much of the power and influence of Bell’s Theorem is due to the apparently simple algebra of the derivation of Bell’s Inequality in B64. But despite its apparent simplicity, I found that, try as I might, I could get no feeling at all, no sense, for how and why Bell’s Theorem worked.

And thus this journey began. Its original intended destination was a conceptual grasp, a feeling, for why this apparently simple yet far-reaching algebraic result works. In the event, the actual destination was not that. Rather, it was the conclusion that the reason I could get no grasp, no feeling, for why Bell’s Theorem works, is that in fact, it doesn’t.

*Outline*

The remainder of this article contains sections under the following headings:

1*. Description of the Bohm thought experiment, from B51A.
2*. The derivation of Bell’s Inequality, from B64.
3. Bell’s Inequality is the result of an inadmissible substitution.
4. Bell’s Inequality, were its derivation otherwise valid, would apply only to the case of non-local hidden variables, not local ones.
5. Bell’s Theorem, valid or not, addresses only deterministic local hidden variables.
6. The peculiar role of experiment.

*Appendices*

There are four appendices. Appendices 1-3 provide background information for the main argument of the article. Appendix 4 briefly discusses a closely related topic.

Appendix 1*. A brief review of the singlet and triplet states.
Appendix 2*. A sketch of the proof that $P(\mathbf{u}, \mathbf{v}) = -\mathbf{u} \cdot \mathbf{v}$ .

Appendix 3*. The derivation of Bell's Inequality, written in a more transparent notation.
Appendix 4. On "combinatorial" derivations of Bell's Inequality.

*1*. Description of the Bohm thought experiment*

The best way to describe the Bohm thought experiment is simply to quote B51A (Chapter 22, Sec. 16).

> "**16. The Hypothetical Experiment of Einstein, Rosen and Podolsky[8].** We shall now describe the hypothetical experiment of Einstein, Rosen and Podolsky. We have modified the experiment somewhat, but the form is conceptually equivalent to that suggested by them, and considerably easier to treat mathematically.
>
> Suppose that we have a molecule containing two atoms in a state in which the total spin is zero and that the spin of each atom is $\hbar/2$.[9] Roughly speaking, this means that the spin of each particle points in a direction exactly opposite to that of the other, insofar as the spin may be said to have any definite direction at all. Now suppose that the molecule is disintegrated by some process that does not change the total angular momentum. The two atoms will begin to separate and will soon cease to interact appreciably. Their combined spin angular momentum, however, remains equal to zero, because by hypothesis, no torques have acted on the system.
>
> [A paragraph is omitted here that describes what the classical analysis of this situation would be.]
>
> Let us now consider how this experiment is to be described in the quantum theory. Here, the investigator can measure either the $x$, $y$ or $z$ component of the spin of particle No. 1, but not more than one of these components, in any one experiment. Nevertheless, it still turns out as we shall see that whichever component is measured, the results are correlated, so that if the same component of the spin of atom No. 2 is measured, it will always turn out to have the opposite value."

The case referred to in the last sentence of the above quote from Bohm (i.e., "...if the same component of the spin of atom No. 2 is measured...") corresponds to the special case in the next section of $\mathbf{a} = \mathbf{b}$.

B51A goes on to further discuss the recast thought experiment and its relation to the questions discussed by EPR35.

Figure 1 is a schematic diagram of the Bohm thought experiment.

[8] For reasons that are unknown to me, in B51A, as well as in Bohm and Aharonov (1957), the order of names in Einstein, Podolsky and Rosen, a.k.a. EPR, is consistently rearranged as Einstein, Rosen and Podolsky, or ERP. I quote Bohm accurately here but prefer not to use the insertion "[*sic*]", whence this explanatory footnote.

[9] This is the singlet state; Appendix 1 provides a brief review of it.

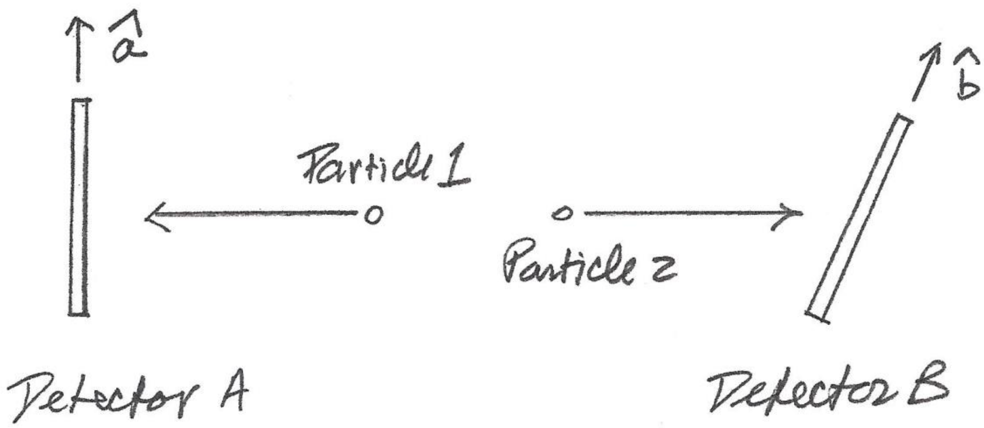


*Figure 1.* Schematic diagram of the Bohm thought experiment. Particles 1 and 2 are spin-½ particles in the singlet state. They move apart toward spin detectors A and B. Unit vectors **a** and **b** are the axes along which the detectors will measure the spins of the respective particles.

*2*. The derivation of Bell's Inequality, as written in B64*[10]

In this section a reprise of the derivation of Bell's Inequality is presented. In this section, no critique will accompany the presentation of the derivation; that will be offered later, in subsequent sections. This reprise is necessary because we will examine certain parts of the derivation in detail. Such detailed examination would be very awkward and inconvenient if it were done entirely with reference to the separate, external text of B64.

B64 defines the symbol $A(\mathbf{n}, \lambda)$ to represent the deterministic result of a measurement by a detector, to be called here A, which is imagined to be, for example, a Stern-Gerlach apparatus oriented along unit vector **n**, while a similar measurement by a detector B is represented by the symbol $B(\mathbf{n}, \lambda)$. The possible values of $A$ and $B$ are $\pm 1$, representing, respectively, spin-up and spin-down. The symbol $\lambda$ represents a hidden variable. Now follows an extended quotation from B64, with some ellipses. However, the equation numbers that appear are my own, not those of B64.

*[Begin quote]*

The result $A$...is...determined by **a** and $\lambda$, and the result $B$ is determined by **b** and $\lambda$, and

[10] The way in which B64 uses the symbols *A*, *B*, **a**, **b** and **c** can be very confusing (*A* vs. **a**, *B* vs. **b, c** as the alternate setting of **b**, etc). This is especially so any time that one tries to *talk* about the paper, but it also can be confusing even when one tries for the first time to *read* or *think* about the paper. Appendix 3 provides the same derivation using a less confusing set of symbols.

$$A(\mathbf{a},\lambda) = \pm 1, \qquad B(\mathbf{b},\lambda) = \pm 1 \quad . \quad (1)$$

The vital assumption [2][11] is that the result *B* for particle 2 does not depend on the setting **a**, of the magnet for particle 1, nor *A* on **b**.

If $\rho(\lambda)$ is the probability distribution of $\lambda$ then the expectation value of the product of the two components $\boldsymbol{\sigma_1} \bullet \mathbf{a}$ and $\boldsymbol{\sigma_2} \bullet \mathbf{b}$[12] is

$$P(\mathbf{a},\mathbf{b}) = \int d\lambda\, \rho(\lambda)\ A(\mathbf{a},\lambda) B(\mathbf{b},\lambda) \quad . \quad (2)$$

This should equal the quantum-mechanical expectation value, which for the singlet state is[13]

$$< \boldsymbol{\sigma}_1 \bullet \mathbf{a} \ \ \boldsymbol{\sigma}_2 \bullet \mathbf{b} > = -\mathbf{a} \bullet \mathbf{b} \quad . \quad (3)$$

But it will be shown that this is not possible.

Some might prefer a formulation in which the hidden variables fall into two sets, with *A* dependent on one and *B* dependent on the other; this possibility is contained in the above, since λ stands for any number of variables and the dependence thereon of *A* and *B* are unrestricted.

[...] There is no difficulty in reproducing, in the form (2), the only features of (3) commonly used in verbal discussions of this problem:

$$P(\mathbf{a},\mathbf{a}) = -P(\mathbf{a},-\mathbf{a}) = -1 \quad (4a)$$
$$P(\mathbf{a},\mathbf{b}) = 0 \ \text{ if } \ \mathbf{a} \bullet \mathbf{b} = 0 \quad (4b)$$

[...] Because $\rho$ is a normalized probability distribution,

$$\int d\lambda\, \rho(\lambda) = 1 \quad (5)$$

and because of the properties (1), *P* in (2) cannot be less than −1. It can reach −1 at **a** = **b** only if

[11] This "[2]" is a reference by B64 to Einstein (1949). It is essential here to understand that Einstein did not doubt the *statistical* inter-dependence of distantly separated results in quantum-mechanical systems, of which the Bohm experiment on the singlet state is one example. What Einstein rejected was a direct, *instantaneous causal* dependence of one measurement upon the results of the other, i.e., spooky action. What Einstein insisted upon was that there must be some as-yet unknown physics which would render such instantaneous causal dependence logically unnecessary. Among other writings by Einstein, the essay of Einstein (1949) from which B64 quotes makes it very evident that this was Einstein's view. What the schema of B64 attempts to do is, among other things, to consider unknown hidden variables $\lambda$ as the hypothetical mediators of the correlations between the detector results, and whether it is possible that such mediators could render spooky action logically unnecessary.

[12] $\boldsymbol{\sigma}$ is the three-dimensional Pauli spin operator, $\mathbf{i}\sigma_x + \mathbf{j}\sigma_y + \mathbf{k}\sigma_z$

[13] Appendix 2 provides a sketch of the derivation of this quantum-mechanical expectation value.

$$A(\mathbf{a},\lambda) = -B(\mathbf{a},\lambda) \quad (6)$$

except at a set of points $\lambda$ of zero probability. Assuming this, (2) can be rewritten

$$P(\mathbf{a},\mathbf{b}) = -\int d\lambda\, \rho(\lambda)\; A(\mathbf{a},\lambda)\, A(\mathbf{b},\lambda)\ . \quad (7)$$

It follows that [if] **c** is another unit vector

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = -\int d\lambda\, \rho(\lambda)\, [\, A(\mathbf{a},\lambda)A(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)A(\mathbf{c},\lambda)] \quad (8a)$$
$$= \int d\lambda\, \rho(\lambda)\, A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)[A(\mathbf{b},\lambda)A(\mathbf{c},\lambda) - 1] \quad (8b)$$

using (1), whence

$$|\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \le \int d\lambda\, \rho(\lambda)\, [1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)]\ . \quad (9)$$

The second term on the right is $P(\mathbf{b}, \mathbf{c})$, whence

$$1 + P(\mathbf{b},\mathbf{c}) \ \ge |\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})\, |\, . \quad (10)$$

*[End quote.]*

Equation (10) is Bell's Inequality.

In B64 there follows a short argument that ends in a counter-example to (10), thus showing that (10) is inconsistent with the quantum-theoretical expectation value for *P*, Equation (3). The counter-example consists of three unit vectors **a**, **b** and **c** all lying in a plane. The unit vectors **a** and **c** are perpendicular to each other, while **b** lies midway between them, that is, 45° from each of them. Then the quantum-mechanical expectation values would be

$$P(\mathbf{a},\mathbf{b}) = -\mathbf{a} \bullet \mathbf{b} = -0.707 \qquad (11a)$$
$$P(\mathbf{a},\mathbf{c}) = -\mathbf{a} \bullet \mathbf{c} = \quad 0 \qquad (11b)$$
$$P(\mathbf{b},\mathbf{c}) = -\mathbf{b} \bullet \mathbf{c} = -0.707 \qquad (11c)$$

which is inconsistent with Bell's Inequality (10):

$$1 + (-0.707) = 0.293 \ \ngeq |\, -0.707 - 0\, | \ = 0.707 \quad (12)$$

It is thus ostensibly proven that inserting local hidden variables $\lambda$ into a quantum-mechanical calculation of the expectation value of a certain observable of the Bohm thought experiment leads to a contradiction with quantum mechanics.

*3. Bell's Inequality is the result of an inadmissible substitution.*

The derivation of Bell's Inequality, Equation (10), depends critically upon substituting Equation (6) into Equation (2) to obtain Equation (7). However, Equation (6) is deduced in the first place by substituting the equation $P(\mathbf{a}, \mathbf{a}) = -1$ from Equations (4a) into Equation (2). Equation (6) thus holds *only for cases covered by the first of Equations (4a), i.e.,* $P(\mathbf{a}, \mathbf{a}) = -1$. Therefore, we may not use Equation (7) in cases for which Equation (6) does not hold, i.e., for cases in which $\mathbf{a} \neq \mathbf{b}$. It follows that for $\mathbf{b} \neq \mathbf{c}$, the second term on the right-hand side of (9) is not equal to $P(\mathbf{b}, \mathbf{c})$ (Equation [2]); and thus Equation (10), Bell's Inequality, is not derived for general **a**, **b** and **c**, only for the case $\mathbf{a} = \mathbf{b} = \mathbf{c}$. [14] The inadmissible use of Equation (7) for cases excluded by its derivation ($\mathbf{a} \neq \mathbf{b}$) is the ultimate source of the contradiction between Bell's Inequality and the quantum theory of the Bohm experiment. The counter-example, Equations (11) and (12), is drawn from the set of cases ($\mathbf{a} \neq \mathbf{b}$) that are not covered by the derivation. It is thus no surprise that it contradicts Equation (10).

This becomes more clear when we write Equations (1) and (6) in the following form, reflecting the quantum theory of the singlet state. (Single primes are used here in equation numbers to indicate correspondence, albeit with a difference, to similarly numbered equations in Section 2 above.)

$$A = B \qquad \text{when} \quad \mathbf{a} \cdot \mathbf{b} = -1 \quad (1\text{a}')$$

$$A = -B \qquad \text{when} \quad \mathbf{a} \cdot \mathbf{b} = 1 \quad (1\text{b}')$$

$$A = B \ \text{or}^{15} \ A = -B \qquad \text{when} \quad |\mathbf{a} \cdot \mathbf{b}| \neq 1 \quad (1\text{c}').^{16,17}$$

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[14] The matter of what cases are actually covered by Equation (10) is rather tangled. Untangling it is a distraction, and so it is relegated here to a footnote. Equation (6) is derived by using $P(\mathbf{a}, \mathbf{a}) = -1$ from Equation (4a). Thus, as previously noted, Equation (7) is *explicitly* derived in B64 *only* for the case of $\mathbf{a} = \mathbf{b}$ (indeed, the sentence in which Equation [6] is embedded says as much). Thus Equation (10), Bell's Inequality, is in turn also explicitly derived only for the case of $\mathbf{a} = \mathbf{b}$. It is not difficult to use $P(\mathbf{a}, -\mathbf{a}) = 1$, also from Equations (4a), to show that for the case of $\mathbf{a} = -\mathbf{b}$, it follows that $A(\mathbf{a}, \lambda) = B(-\mathbf{a}, \lambda)$; and from thence to derive Bell's Inequality, Equation (10), *along a path that does not begin with Equation (7)*; rather, the path begins with a variant of (7), one which lacks the minus sign before the integral. It is only application of the absolute value operator which brings the two paths together at the end to yield (10) for both cases. We forego providing the second derivation here; suffice it to note that substitution easily shows that Bell's Inequality holds for the three cases $\mathbf{a} = \mathbf{b} = \mathbf{c}$, $\mathbf{a} = -\mathbf{b} = \mathbf{c}$, and $\mathbf{a} = \mathbf{b} = -\mathbf{c}$, though it is not derived in B64 for the latter two relations. That Equation (10) does not hold for *general* **a**, **b**, and **c** is the subject of this section.

[15] This is an exclusive "or". We know very little about *A* and *B*, but they are clearly intended in B64 to be single-valued functions of their arguments.

[16] From Equations (1a′) and (1b′) it may be observed that *A* and *B* are not functions representing solely the detectors. Rather, they represent some amalgam of the properties of the detectors with those of the particles, thus yielding the observation result.

Restating the matter through the clear lens of Equations (1′): Bell's Inequality is derived in B64 only from (1b′) and thus, as derived, applies only in that case. As discussed in footnote 14, Bell's Inequality can be derived also, through a somewhat different path, from (1a′). Bell's Inequality is not derived in B64, and cannot be derived at all, from (1c′), which is the most general case. The absence of (1c′) from the derivation is the source of the contradiction between Bell's Inequality and the quantum theory of the Bohm experiment. The counter-example of Equations (11) and (12) is drawn from (1c′), for which Bell's Inequality was not derived. The use of Equation (10) beyond the assumptions of its derivation is the first of the mathematical errors referred to in the abstract.

It is instructive to re-perform the derivation of Bell's Inequality, this time without substituting (6) into (2), *i.e.*, without being restricted to the cases of Equations (1a′) and (1b′). We begin with the original Equation (2) and work the subsequent algebra explicitly:

$$P(\mathbf{a},\mathbf{b}) = \int d\lambda\, \rho(\lambda)\; A(\mathbf{a},\lambda)\, B(\mathbf{b},\lambda)\ . \quad (2)$$

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = \int d\lambda\, \rho(\lambda)\, [\, A(\mathbf{a},\lambda)B(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)B(\mathbf{c},\lambda)]. \quad (8a')$$

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a},\lambda)B(\mathbf{b},\lambda) \left[1 - \frac{A(\mathbf{a},\lambda)B(\mathbf{c},\lambda)}{A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)}\right]\ .$$

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a},\lambda)B(\mathbf{b},\lambda) \left[1 - \frac{B(\mathbf{c},\lambda)}{B(\mathbf{b},\lambda)}\right]\ .$$

$$P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)[1 - B(\mathbf{b},\lambda)B(\mathbf{c},\lambda)]\ . \quad (8b')$$

$$|\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \le \int d\lambda\, \rho(\lambda)\, |1 - B(\mathbf{b},\lambda)B(\mathbf{c},\lambda)|\ . \quad (8c')$$

The expression inside the absolute value operator on the right-hand side of (8c′) is always non-negative; thus

$$|\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \le \int d\lambda\, \rho(\lambda)\, [1 - B(\mathbf{b},\lambda)B(\mathbf{c},\lambda)]\ . \quad (9')$$

[17] The reference in B64 to "the only features of [(3)] commonly used in verbal discussions of this problem" is an arbitrary restriction. The feature of (3) expressed in Equation (1c′) is as well-known, and as much of the essence of the problem, as those expressed in (1a′) and (1b′).

It is essential to observe here that Equation (9′) is *not* the same as (9), despite its apparent similarity. In (9), the symbol $A$ represents a function that characterizes the result in *either* detector. In (9′) the symbols $A$ and $B$ refer to the results obtained in the A and B detectors *respectively*. That is to say, by not inappropriately using (6), Equation (9′) preserves the separate identity of each detector and covers all three of the cases of Equations (1′).

Rearranging (9′) into a form convenient for comparison with Equation (10), Bell's Inequality, we obtain

$$1 - \int d\lambda\, \rho(\lambda)\, B(\mathbf{b},\lambda) B(\mathbf{c},\lambda) \;\geq |\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \,. \quad (10')$$

What is the second term on the left side of (10′), the integral? It evidently is not $P(\mathbf{b}, \mathbf{c})$ (Equation [2]), because it involves the B detector alone. It is the correlation, as a function of $\lambda$, of the results of two hypothetical unequal settings of the B detector.

*These two unequal settings of the B detector are cases that cannot both be realized in the observation of a single pair of particles.* Because the unequal settings of detector B cannot be realized in the observation of a single pair of particles, there are no experiments that could be carried out to make a direct empirical estimate of what the value of the integral on the left side of (10′) might be.

Moreover, we have no specific ideas concerning the general functional dependence of $B(\mathbf{n}, \lambda)$ on $\lambda$, and therefore we have no way to calculate a theoretical (i.e., predicted) value for the integral on the left-hand side of (10′).

Thus, Equation (10′), the inequality that is derived for general **a**, **b** and **c** without the restrictive Equation (6), cannot be compared either to experiment or to theory.

Christian [2018, Sections 4.1 and 4.2], by a different path of analysis, has demonstrated a similar point with respect to the CHSH form of Bell's Inequality. The CHSH Inequality is derived similarly to Bell's Inequality, but without the use of Equation (6).

To work Equation (8c′) into a form that can be compared to theory and experiment for general **a**, **b** and **c**, the only thing we can do is to recognize that the expression inside the absolute value sign on the right-hand side of (8c′) is bounded above by 2. Equation (8c′) then becomes

$$|\, P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq 2 \;. \quad (9a')$$

This result, (9a′), is obtained by carrying forward the basic schema of B64 but in an algebraically consistent manner. For general **a**, **b** and **c,** (9a′) is consistent with both the quantum theory of the Bohm experiment and the experimental results. Of course, Equation (9a′) is not proof, or even evidence, that hidden variables $\lambda$ exist.

*4. Bell's Inequality, were its derivation otherwise valid, would apply only to the case of non-local hidden variables, not local ones.*

Let us for the moment set aside the error described in Section 3 above and continue our analysis of Bell's Inequality as if it were valid for all three cases of Equations (1′). In doing so we shall find that there is another error in Bell's Theorem, independent of the one discussed in the previous section.

In the short paragraph following Equation (3), Bell writes:

> Some might prefer a formulation in which the hidden variables fall into two sets, with *A* dependent on one and *B* dependent on the other; this possibility is contained in the above [Equation (2)], since λ stands for any number of variables and the dependence thereon of *A* and *B* are unrestricted.

Let us try the formulation which "some might prefer," to see what effect it has on the result.

We begin with a modification of Equation (2). (In this section double primes are used to indicate correspondence, albeit with a difference, to similarly numbered earlier equations.)

$$P(\mathbf{a},\mathbf{b}) = \int\int \rho(\lambda_1,\, \lambda_2)\; A(\mathbf{a},\lambda_1)B(\mathbf{b},\lambda_2)\; d\lambda_1\, d\lambda_2 \;. \quad (2'')$$[18]

Here, we have assumed that the local hidden variable $\lambda_i$ is associated with the *i*th particle and not with the corresponding detector. This is the case of principal interest; however, later in this section we will also briefly consider the case in which the local hidden variable may be associated with the detector, or with both the particle and the detector.

We must also modify Equation (6) to become (6″):

$$A(\mathbf{a},\lambda_1) = -B(\mathbf{a},\lambda_2)\;\;, \quad (6'')$$

Inserting (6″) in (2″) in the manner of B64 leads to a modified version of Equation (7):

$$P(\mathbf{a},\mathbf{b}) = -\int\int \rho(\lambda_1,\, \lambda_2)\; A(\mathbf{a},\lambda_1)A(\mathbf{b},\lambda_2)\; d\lambda_1\, d\lambda_2 \;. \quad (7'')$$

We now follow the algebraic steps of Equations (8a) and (8b) in Section 1 above, and derive an equation corresponding to Equation (9):

[18] The density function $\rho(\lambda_1,\, \lambda_2)$ in Equation (2'') includes the special case of $\rho = \rho_1(\lambda_1)\rho_2(\lambda_2)$.

$$| P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq \int \int \rho(\lambda_1, \lambda_2)\, d\lambda_1 \,[1 - A(\mathbf{b},\lambda_2)A(\mathbf{c},\lambda_2)]\; d\lambda_2 \,. \quad (9'')$$

This equation may be simplified as

$$| P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq \int \rho(\lambda_2)\; [1 - A(\mathbf{b},\lambda_2)A(\mathbf{c},\lambda_2)]\, d\lambda_2 \,. \quad (9\mathrm{a}'')$$

Equation (9a″) is then rearranged for easy comparison to Equation (10) as

$$1 - \int \rho(\lambda_2)\; A(\mathbf{b},\lambda_2)A(\mathbf{c},\lambda_2)\, d\lambda_2 \;\geq | P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \;. \quad (10'')$$

Now we see that we have encountered a difficulty similar to that of the previous section, even though this time we have followed B64 and allowed the use of Equation (6″) in (2″) to create (7″). The difficulty is that the second term on the left side of Equation (10″) is not equal to *P*(**b**, **c**), Equation (2″), because the integral is over $\lambda_2$ only, not over $\lambda_1$ and $\lambda_2$ together.

There is no experiment that we can conduct in which the detector associated with particle 2 is to be simultaneously in two different orientations, **b** and **c**. Therefore, we can make no experimental estimate of the value of the integral that we can compare with the other terms. Also, we know nothing about the dependence of *A* on $\lambda_1$. Therefore, we cannot theoretically predict the value of the integral so that it can be compared to the other terms in (10″). Thus we cannot compare (10″) either to theory or to experiment.

On the other hand, if there is only a single non-local $\lambda$, such that $\lambda_1 = \lambda_2 = \lambda$, then (10″) is reduced to (9), which becomes (10). Equation (10), Bell's Inequality, can be and of course has been compared to both theory and experiment and famously matches neither.

Thus we see that Equation (10), Bell's Inequality, if it were otherwise consistent and were to be compared to theory and experiment, would exclude only hypothetical *non-local* hidden variables, where $\lambda$ is identically the same in the arguments of functions *A* and *B*.

If we were to associate the local hidden variable with the detector instead of with the particle, or with both the detector and the particle, then *A*(**c**, $\lambda_2$) would become *A*(**c**, $\lambda_3$), or become some other similar expression for *A*(**c**) as a function of a more complicated set of arguments $\lambda_i$. The impossibility of the comparison of the inequality (10″) to either theory or experiment would not be relieved.

What, then, of Bell's assertion that

> "...a formulation in which the hidden variables fall into two sets, with *A* dependent on one and *B* dependent on the other...is contained in the above [Equation (2)], since $\lambda$ stands for any number of variables and the dependence thereon of *A* and *B* are unrestricted"?

This assertion seems to amount to the suggestion that $\lambda$ is to be understood as a vector $\lambda = (\lambda_1, \lambda_2)$[19]; that $A(\mathbf{n}, \lambda)$ is to be understood in the sense of $A(\mathbf{n}, \lambda_1, \lambda_2)$, and similarly for $B$; that, however, $\lambda_2$ is to be understood as an argument of $A$ only in a purely formal sense, but not having any actual effect on $A$; while $\lambda_1$ is to be understood as a correspondingly purely formal argument of $B$, lacking any actual effect on $B$. That notational approach has resulted in confusion and error; this is the second of the mathematical errors referred to in the abstract. Those who might have preferred the more explicit formulation used in Equation (2″) would have been right to do so.

*5. Bell's Theorem, valid or not, addresses only deterministic local hidden variables.*

The formulation of the case of hidden variables used in B64, Equation (2), assumes deterministic hidden variables. That is to say, $A(\mathbf{n}, \lambda)$ is a single-valued function of its arguments, and similarly for $B(\mathbf{n}, \lambda)$. B64 does not address the case in which hypothetical local hidden variables do not deterministically yield the result of the observation, but rather contribute to the local generation of the local probability distribution from which the observation result is ultimately drawn. Given that the orthodox interpretation of quantum mechanics already holds that when more than one outcome is possible, the result of an observation is drawn randomly from a probability distribution, this would be a very conservative form for a hypothesis of local hidden variables to take.

EPR would have rejected such a scenario, as violating both the first and the second of their requirements for a complete theory, namely, precisely defined values for all observables at all times, and predictability in principle. However, we are not required to adopt an all-or-nothing point of view with regard to EPR's three objections. If we are prepared to consider the case in which unknown properties of the particles and/or the detectors, and unknown processes in the interactions of the particles with the detectors, influence the local probability distribution for the observation but do not uniquely determine the outcome, then even if Bell's Inequality and its application to the Bohm experiment were algebraically consistent, it still would not be an airtight proof of spooky action, for the simple reason that it addresses only the deterministic scenario upon which Einstein insisted, and does not address non-deterministic scenarios.

This last point should perhaps not be taken as a criticism of B64 itself, as the title of the paper suggests that it is intended only as a response to the three bundled objections of EPR35. However, whatever the paper's original intention, it has come to be regarded as a definitive general proof of non-locality, not only as a proof of the inconsistency of quantum mechanics with the bundled three-part position taken by EPR (a position often referred to as "Einstein locality").

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[19] To be more precise, it is the suggestion that $\lambda = (\lambda_1, \lambda_{2,} ..., \lambda_N)$, where there may be any number N of components $\lambda_i$ . For our purposes, N=2 is sufficient.

*6. The curious role of experiment*

> "For those things which agree in negation alone, or in what they have not, in truth agree in nothing." Spinoza (1677)

Experimental realizations of Bohm's thought experiment have played a very important role in the acceptance of Bell's Theorem. Bell's Theorem and the experimental realizations are in "agreement" in the negative sense that the experimental results do not satisfy the CHSH Inequality. For this reason, the argument of Bell's Theorem has been considered to be verified by experiment.

This is a very forgiving position for a theoretical argument to be in, namely, to be considered correct for predicting not what will be observed, but rather what will fail to be observed. In this case, the prediction of what will fail to be observed benefits from the yet further forgiving characteristic of having been calculated from unspecified hypothetical variables and processes.

Bell's Theorem does indeed predict the right experimental result, namely, that Bell's Inequality in the form of the CHSH Inequality predicts the wrong experimental result. However, it does so for wrong reasons, in particular, because of errors in the mathematical logic that carries its assumptions into its predictions. The "agreement" of Bell's Theorem with experiment is no agreement at all.

*APPENDICES*

*Appendix 1*. Brief review of the singlet and triplet states.*

Most quantum mechanics textbooks cover the material in this appendix. This appendix will loosely follow the notation used by Bohm in B51A, Section 17.9. (I say "loosely" because B51A does not use the Dirac notation.)

Consider a pair of particles, each of spin ½. Consider the operator $\boldsymbol{S_{12}}$, the spin observable for this system:

$$\boldsymbol{S_{12}} = \boldsymbol{S_1} + \boldsymbol{S_2} = \frac{\hbar}{2}(\boldsymbol{\sigma_1} + \boldsymbol{\sigma}_2) \ , \quad (\mathrm{A1}-1)$$

where $\boldsymbol{\sigma}$ is the three-dimensional Pauli spin operator, $\mathbf{i}\sigma_x + \mathbf{j}\sigma_y + \mathbf{k}\sigma_z$, and the operators with subscripts 1 and 2 operate only on particle 1 and particle 2, respectively.

The four eigenvectors of $S_{12z}$, the $z$ component of $\boldsymbol{S}_{12}$, are

$$\psi_a = |++>, \quad \psi_b = |+->, \quad \psi_c = |-+> \quad \text{and} \quad \psi_d = |--> \quad , \quad (\text{A1}-2)^{20}$$

where the + and – signs correspond, respectively, to observations of spin-up and spin-down along the *z* axis, and the four eigenvectors correspond, respectively, to the eigenvalues $\hbar$, 0, 0 and $-\hbar$ for $S_{12z}$ . A linear combination of $\psi_b$ and $\psi_c$ is also an eigenvector of $S_{12z}$ corresponding to an eigenvalue of 0.

We can also consider the operator $S_{12}{}^2$, which is

$$S_{12}{}^2 = (\boldsymbol{S_1} + \boldsymbol{S_2})^2 = S_1{}^2 + S_2{}^2 + 2\boldsymbol{S_1} \cdot \boldsymbol{S_2} = \frac{3}{2}\,\hbar^2 + 2\boldsymbol{S_1} \cdot \boldsymbol{S_2} \qquad (\text{A1}-3)$$

This operator, $S_{12}{}^2$, also has four eigenvectors. Two of them are identical to the eigenvectors given above for the operator $S_{12z}$. These are $\psi_a$ and $\psi_d$ above, both corresponding to the eigenvalue for $S_{12}{}^2$ of $2\hbar^2$. The other two eigenvectors of $S_{12}{}^2$ are

$$\psi_1 = \frac{1}{\sqrt{2}}(\psi_b + \psi_c) \quad \text{and} \quad \psi_2 = \frac{1}{\sqrt{2}}(\psi_b - \psi_c) \ . \qquad (\text{A1}-4)$$

The eigenvalue of $S_{12}{}^2$ associated with $\psi_1$ is $2\hbar^2$, while the eigenvalue associated with $\psi_2$ is 0.

Thus, $\psi_a$, $\psi_1$, $\psi_2$ and $\psi_d$ are all eigenvectors of both of the operators $S_{12z}$ and $S_{12}{}^2$. Eigenvectors $\psi_a$, $\psi_d$, and $\psi_1$ are referred to collectively as the triplet state, while $\psi_2$ is called the singlet state.

The states $\psi_b$ and $\psi_c$ are not eigenvectors of $S_{12}{}^2$.

Eigenvector $\psi_2$ is the subject of the Bohm experiment. In state $\psi_2$, while $S_{12z}$ has a definite value of zero, neither of its two parts, $S_{1z}$ and $S_{2z}$, has a definite value prior to measurement. *When measured along the same direction* (the *z* axis, in this particular case), these two spins of particles 1 and 2 will always be opposite to each other, as expressed in Equation (6), or Equation (1b′), but which spin result (up or down) will be observed in which detector cannot be specified in advance.

*Appendix 2**. A sketch of the proof that $P(\mathbf{u}, \mathbf{v}) = -\mathbf{u} \cdot \mathbf{v}$ .

Like Appendix 1, this appendix loosely follows the notation of B51A. To avoid confusion between the subscripts *a*, *b*, *c*, *d* of Appendix 1 and the unit vectors **a**, **b** and **c** of B64, in this appendix the unit vectors describing the settings of detectors A and B respectively are named **u** and **v** rather than **a** and **b**.

[20] The subscripts *a*, *b*, *c* and *d* are used in B51A, and thus in this appendix, to enumerate eigenstates. This use of *a*, *b* and *c* should not be confused with the **a**, **b** and **c** unit vectors of B64. There is no notational relation between them.

In bra-ket and operator notation, $P$ (**u**, **v**) may be written as

$$P(\mathbf{u},\mathbf{v}) = <\psi_2|(\mathbf{u}\cdot\boldsymbol{\sigma_1})(\mathbf{v}\cdot\boldsymbol{\sigma_2})|\psi_2> \quad , \qquad \text{(A2}-1)$$

where $\psi_2$ is the singlet state, as discussed in Appendix 1, and $\boldsymbol{\sigma}$ is the three-dimensional Pauli spin operator, $\mathbf{i}\sigma_x + \mathbf{j}\sigma_y + \mathbf{k}\sigma_z$. Here, $\boldsymbol{\sigma_1}$ operates on particle 1, while $\boldsymbol{\sigma_2}$ operates on particle 2. The two operators are transparent to each other. For a given pair of detector settings **u** and **v**, $P$ is the expectation value of the product of the two spin results, where spin-up in each detector is assigned +1, and spin-down is assigned −1. Another way of saying this is that $P$ is the average value of the product of the observed spins, divided by $\hbar^2/4$.

Implicitly performing the dot product operation within each operator, we can rewrite (A2−1) as

$$P(\mathbf{u},\mathbf{v}) = <\psi_2|\sigma_u\sigma_v|\psi_2> \quad , \qquad \text{(A2}-2)$$

Recalling from (A1−4) above that $\psi_2$ is a linear combination of $\psi_b$ and $\psi_c$, we expand (A2−2) as

$$P(\mathbf{u},\mathbf{v}) = \frac{1}{2}(<\psi_b| - <\psi_c|)\,\sigma_u\sigma_v\,(|\psi_b> - |\psi_c>) \quad , \qquad \text{(A2}-2)$$

which is further expanded as

$$P(\mathbf{u},\mathbf{v}) = \frac{1}{2}\,[<\psi_b|\sigma_u\sigma_v|\psi_b> - <\psi_c|\sigma_u\sigma_v|\psi_b> - <\psi_b|\sigma_u\sigma_v|\psi_c> + <\psi_c|\sigma_u\sigma_v|\psi_c>]\ , \qquad \text{(A2}-3)$$

Now it is convenient to introduce matrix forms for the operator $\sigma_u$ and the singlet state $\psi_2$. The matrix form for $\psi_2$ (A1 − 4) is written as (B51A, Sec. 17.9, p. 399 )

$$\psi_2 = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\} \quad , \qquad \text{(A2}-4)$$

where the first term inside the curly braces represents $\psi_b = |+->$ while the second term represents $\psi_c = |-+>$. The operator $\sigma_u$ is written in matrix form as

$$\sigma_u = \begin{pmatrix} u_z & u_x - iu_y \\ u_x + iu_y & -u_z \end{pmatrix} \qquad \text{(A2}-5)$$

where $u_x, u_y$ and $u_z$ are the three components of the unit vector **u**, and similarly for $\sigma_v$ (see, e.g., B51A, p. 394, Equation [28]).

We perform the evaluation beginning with the first term inside the square brackets on the left of Equation (A2−3). For the first term, we will carry out the evaluation in detail, so that the use of the notation will be fully illustrated. Recalling that each operator operates only on one particle, we have

$$\sigma_u|\psi_b> = \sigma_u \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_z \\ u_x + iu_y \end{pmatrix} \quad \text{(A2 − 6a)}$$

and

$$\sigma_v|\psi_b> = \sigma_v \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_x - iv_y \\ -v_z \end{pmatrix} \ . \quad \text{(A2 − 6b)}$$

These are combined into a product as

$$\sigma_u\sigma_v|\psi_b> = \sigma_u \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_v \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_z \\ u_x + iu_y \end{pmatrix} \begin{pmatrix} v_x - iv_y \\ -v_z \end{pmatrix} \quad \text{(A2 − 7)}$$

The bra $< \psi_b|$ is applied as two row vectors:

$$< \psi_b|\,\sigma_u\sigma_v|\psi_b> = \begin{matrix} (1\ 0) \\ (0\ 1) \end{matrix}\, \sigma_u \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_v \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{matrix} (1\ 0) \\ (0\ 1) \end{matrix} \begin{pmatrix} u_z \\ u_x + iu_y \end{pmatrix} \begin{pmatrix} v_x - iv_y \\ -v_z \end{pmatrix} \quad \text{(A2 − 8)}$$

In the above expression, on the right-hand side, the top row vector multiplies the left column vector, the bottom row vector multiplies the right column vector, and the two resulting scalars are combined as a product. When this multiplication is carried out, we obtain

$$< \psi_b|\,\sigma_u\sigma_v|\psi_b> = -u_z v_z \ . \quad \text{(A2 − 9)}$$

Applying the same method, the fourth term inside the square brackets of (A2−3) is evaluated as

$$< \psi_c|\,\sigma_u\sigma_v|\psi_c> = -u_z v_z \ . \quad \text{(A2 − 10)}$$

The second term in the square brackets in (A2−3) is evaluated as

$$-< \psi_c|\sigma_u\sigma_v|\psi_b> = -\left(u_x v_x + u_y v_y + iu_y v_x - iu_x v_y\right) \ . \quad \text{(A2 − 11)}$$

The third term inside the square brackets in (A2−3) is the complex conjugate of the second term:

$$-<\psi_b|\sigma_u\sigma_v|\psi_c> = -\left(u_x v_x + u_y v_y - iu_y v_x + iu_x v_y\right) \ . \ (\mathrm{A2}-12)$$

Inserting these four partial results, namely, (A2–9), (A2–10), (A2–11) and (A2–12), into (A2–3), we obtain

$$P(\mathbf{u},\mathbf{v}) = -(u_x v_x + u_y v_y + u_z v_z) \ , \quad (\mathrm{A2}-13)$$

which we recognize as

$$P(\mathbf{u},\mathbf{v}) = -\mathbf{u} \cdot \mathbf{v} \ . \quad (\mathrm{A2}-14)$$

*Appendix 3*. The derivation of Bell's Inequality, written in a more transparent notation*

This appendix contains the same derivation as that given in Section 2, except recast in what I hope is a less confusing notation.

To make it easy to compare the contents of this appendix with Section 2, the language of Section 2 is re-used with only a few changes. Please note that unlike in Section 2, in this appendix the indentation of the text from both margins signifies something less than a *verbatim* quote from B64. Also, in this appendix the footnotes of Section 2 have been dropped. The equation numbers in this appendix correspond to those of Section 2, distinguished only by the use of "A3–" (for Appendix 3) in the equation number.

We begin by defining the symbol $M_1(\mathbf{n_1}, \lambda)$ to represent the result of a measurement by detector 1 oriented along unit vector $\mathbf{n_1}$, while a similar measurement by detector 2 is represented by the symbol $M_2(\mathbf{n_2}, \lambda)$. The symbol $\lambda$ represents a local hidden variable.

> *[Begin derivation]*
>
> The result $M_1$...is...determined by $\mathbf{n_1}$ and $\lambda$, and the result $M_2$ is determined by $\mathbf{n_2}$ and $\lambda$, and
>
> $$M_1(\mathbf{n_1},\lambda) = \pm 1, \qquad M_2(\mathbf{n_2},\lambda) = \pm 1 \ . \quad (\mathrm{A3}-1)$$
>
> The vital assumption is that the result $M_2$ for particle 2 does not depend on the setting $\mathbf{n_1}$, of the magnet for particle 1, nor $M_1$ on $\mathbf{n_2}$.
>
> If $\rho(\lambda)$ is the probability distribution of $\lambda$ then the expectation value of the product of the two components $\boldsymbol{\sigma_1} \bullet \mathbf{n_1}$ and $\boldsymbol{\sigma_2} \bullet \mathbf{n_2}$ is
>
> $$P(\mathbf{n_1},\mathbf{n_2}) = \int d\lambda\, \rho(\lambda)\ M_1(\mathbf{n_1},\lambda) M_2(\mathbf{n_2},\lambda) \ . \quad (\mathrm{A3}-2)$$

This should equal the quantum-mechanical expectation value, which for the singlet state is

$$< \boldsymbol{\sigma}_1 \bullet \mathbf{n_1} \;\; \boldsymbol{\sigma}_2 \bullet \mathbf{n_2} > = -\mathbf{n_1} \bullet \mathbf{n_2} \; . \quad \text{(A3} - 3)$$

But it will be shown that this is not possible.

Some might prefer a formulation in which the hidden variables fall into two sets, with $M_1$ dependent on one and $M_2$ dependent on the other; this possibility is contained in the above, since λ stands for any number of variables and the dependence thereon of $M_1$ and $M_2$ are unrestricted.

[...] There is no difficulty in reproducing, in the form (A3−2), the only features of (A3−3) commonly used in verbal discussions of this problem:

$$P(\mathbf{n}, \mathbf{n}) = -P(\mathbf{n}, -\mathbf{n}) = -1 \quad \text{(A3} - 4\text{a)}$$

where **n** is any unit vector, and

$$P(\mathbf{n_1}, \mathbf{n_2}) = 0 \;\text{ if }\; \mathbf{n_1} \bullet \mathbf{n_2} = 0 \quad \text{(A3} - 4\text{b)}$$

[...] Because $\rho$ is a normalized probability distribution,

$$\int d\lambda \, \rho(\lambda) = 1 \quad \text{(A3} - 5)$$

and because of the properties (A3−1), $P$ in (A3−2) cannot be less than −1. It can reach −1 at $\mathbf{n_1} = \mathbf{n_2} = \mathbf{n}$ only if

$$M_1(\mathbf{n}, \lambda) = -M_2(\mathbf{n}, \lambda) \quad \text{(A3} - 6)$$

except at a set of points $\lambda$ of zero probability. Assuming this, (A3−2) can be rewritten

$$P(\mathbf{n_1}, \mathbf{n_2}) = -\int d\lambda \, \rho(\lambda) \;\; M_1(\mathbf{n_1}, \lambda) \, M_1(\mathbf{n_2}, \lambda) \; . \quad \text{(A3} - 7)$$

It follows that [if] $\mathbf{n_2}'$ is another unit vector

$$\begin{aligned} P(\mathbf{n_1}, \mathbf{n_2}) - \; & P(\mathbf{n_1}, \mathbf{n_2}') \\ & = -\int d\lambda \, \rho(\lambda) \, [\, M_1(\mathbf{n_1}, \lambda) \, M_1(\mathbf{n_2}, \lambda) - M_1(\mathbf{n_1}, \lambda) \, M_1(\mathbf{n_2}', \lambda)] \quad \text{(A3} - 8\text{a)} \\ & = \int d\lambda \, \rho(\lambda) \, M_1(\mathbf{n_1}, \lambda) \, M_1(\mathbf{n_2}, \lambda) [M_1(\mathbf{n_1}, \lambda) \, M_1(\mathbf{n_2}', \lambda) - 1] \quad \text{(A3} - 8\text{b)} \end{aligned}$$

using (A3−1), whence

$$| P(\mathbf{n_1}, \mathbf{n_2}) - P(\mathbf{n_1}, \mathbf{n_2}')| \leq \int d\lambda\, \rho(\lambda)\, [1 - M_1(\mathbf{n_2}, \lambda)\, M_1(\mathbf{n_2}', \lambda)] \ . \quad \text{(A3} - 9)$$

The second term on the right is $P(\mathbf{n_1}, \mathbf{n_2}')$, whence

$$1 + P(\mathbf{n_2}, \mathbf{n_2}') \ \geq | \, P(\mathbf{n_1}, \mathbf{n_2}) - P(\mathbf{n_1}, \mathbf{n_2}') \, | \, . \quad \text{(A3} - 10)$$

*[End derivation.]*

Equation (A3−10) is Bell's Inequality.

In B64 there follows a short argument that ends in a counter-example to (A3−10), thus showing that (A3−10) is inconsistent with the quantum-theoretical result, Equation (A3−3). The counter-example consists of three unit vectors $\mathbf{n_1}$, $\mathbf{n_2}$ and $\mathbf{n_2}'$ all lying in a plane. The unit vectors $\mathbf{n_1}$ and $\mathbf{n_2}'$ are perpendicular to each other, while $\mathbf{n_2}$ lies midway between them, that is, 45° from each of them. Then the quantum-mechanical expectation values would be

$$P(\mathbf{n_1}, \mathbf{n_2}) \ = \ -\mathbf{n_1} \bullet \mathbf{n_2} \ = \ -0.707 \qquad \text{(A3} - 11\text{a)}$$
$$P(\mathbf{n_1}, \mathbf{n_2}') = \ -\mathbf{n_1} \bullet \mathbf{n_2'} = \quad 0 \qquad \text{(A3} - 11\text{b)}$$
$$P(\mathbf{n_2}, \mathbf{n_2}') \ = \ -\mathbf{n_2} \bullet \mathbf{n_2}' = \ -0.707 \qquad \text{(A3} - 11\text{c)}$$

which is obviously inconsistent with Bell's Inequality (A3−10):

$$1 + (-0.707) = 0.293 \ \ngeq | -0.707 - 0 \, | \ = 0.707 \quad \text{(A3} - 12)$$

It is thus ostensibly proven that inserting local hidden variables into the quantum-mechanical calculation for the Bohm thought experiment leads to a contradiction with quantum mechanics.

*Appendix 4. On "combinatorial" derivations of Bell's Inequality*

Distinct from the original derivation of Bell's Inequality in B64, there is a class of derivations of Bell's Inequality which I call the "combinatorial" derivations. Bell (1981; reprinted in Bell, 1987) refers the origins of this class of derivations to Wigner and d'Espagnat. Sakurai (1994) refers them to Wigner. It seems necessary in the context of this article to comment on these, though the discussion here must be brief and thus does not do justice to the subject. For details of the combinatorial derivations, please see the references.

The combinatorial derivations are set in a variety of story lines. The most famous and the most fanciful is that of Bertlmann's socks (Bell, 1981; see also Baggott, 1992). Other variations that I have encountered are those of Sakurai (1994), Rieffel and Polak (2011), and Nielsen and Chuang (2010). Presumably there are others, possibly many others. Although there are some minor variations in the substance of the assumptions that are used, all of the story lines clothe essentially the same argument.

The essence of the combinatorial derivations is to set up some hypothetical system that satisfies Einstein's objections to non-commuting observables and unpredictability by giving some set of objects definite values of one or more binary variables (i.e., observables) prior to being observed. In each hypothetical system the general framework of Bohm's two-particle experiment can be recognized, even when, as in the example of Bertlmann's socks, the system is not a narrative of particles and spins. From the characteristics of the hypothetical system, Bell's Inequality, or, equivalently, the CHSH Inequality, is derived by counting the different possible combinations of outcomes of observations made on the objects and relating this to the expected number of times each outcome would be obtained. It is then shown that the derived inequality (Bell's, or the CHSH, as the case may be) is in conflict with both the theory and the results of the Bohm experiment, thus ostensibly falsifying Einstein's objections to the completeness of quantum mechanics.

This approach is different from that of B64. The development of Bell's Inequality in B64 in effect claims (incorrectly, as we have seen) to stipulate the quantum theory of the Bohm experiment, and then to develop the implications of adding hypothetical local variables to the quantum theory. One might say that the combinatorial derivations, which arrive at the same inequality as B64, are more accurately self-aware in regard to what they are actually assuming, in that they do not stipulate the quantum theory – quite the opposite. Moreover, I am not aware of mathematical errors *per se* in the combinatorial derivations, unlike in B64.

As stated above, each of the combinatorial derivations begins by describing a specific, explicitly non-quantum system that is designed to appeal to "common sense" and does not exhibit, indeed, cannot exhibit, the correlations between detectors that are expected in the Bohm experiment. The behavior of this non-quantum system, and especially the ways in which it yields results that are different from the quantum theory of the Bohm experiment, are analyzed. This is interesting and instructive. But comparing and contrasting *hypothetical, specific, explicitly non-quantum* systems with the quantum theory of the Bohm experiment does not constitute a proof that it is impossible that *any* quantum system could exist in which the measured values of non-commuting observables might emerge from as-yet unknown local properties and processes in the particle and its interaction with the local detector. In other words, such comparison and contrast does not constitute a proof that it is impossible that the physical phenomena described in quantum mechanics as the singlet state can exist without the property of spooky action.

Moreover, with regard to unknown non-deterministic local processes, the combinatorial derivations do not speak. The hypothetical systems considered in the combinatorial derivations are always deterministic. Einstein, of course, objected to non-deterministic processes, though not as much as he objected to spooky action. However, as previously stated, we need not take the point of view that all three of EPR's objections to the completeness of quantum mechanics must stand or fall together.

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